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A point's point of view of stringy geometry

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Abstract: The notion of a "point" is essential to describe the topology of spacetime. Despite this, a point probably does not play a particularly distinguished rôle in any intrinsic formulation of string theory. We discuss one way to try to determine the notion of a point from a worldsheet point of view. The derived category description of D-branes is the key tool. The case of a flop is analyzed and Π-stability in this context is tied in to some ideas of Bridgeland. Monodromy associated to the flop is also computed via Π-stability and shown to be consistent with previous conjectures.

Keywords: D-branes, Differential and Algebraic Geometry.

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1. Introduction

One might view the central philosophy of string theory as trying to avoid basing our interpretation of spacetime on the notion of a *point*. Having said that, all our concepts of geometry do appear ultimately to remain firmly wedded to points. It hard not to begin any discussion of spacetime without a topological space, i.e., a set of points.

Compactification produces an excellent laboratory for studying the stringy geometry of space and, in particular, allows us to evade all the thorny issues associated with time. For example, one takes a type II string and compactifies on some Calabi-Yau threefold X to produce an effective theory with four-dimensional spacetime. The geometry of the Calabi-Yau dictates the properties of the conformal field theory associated to the non-linear sigma model of the compactification, which in turn effects the physics in four dimensions. Thus the geometry of X can be analyzed in an inherently stringy way from the uncompactified dimensions. By including boundaries on the worldsheet we may analyze D-branes (at least near the zero-coupling limit of string theory). We can then look at "0-branes", i.e., D-branes which span zero dimensions of X.

It is these 0-branes which form the bridge between the intrinsically stringy notion of the worldsheet and the more classical notion of a point. In this note we discuss a couple of properties of 0-branes to highlight how they are key in describing the topology of the target space.

This idea is far from new. Using 0-branes to probe the target space has been used frequently in the past. To name just a couple of examples, the SYZ conjecture [1] relies on this concept, and in [2, 3] it was used to analyze orbifolds. We refer the reader to [4] for further discussion and references. Reconstructing the target space using D-branes was also recently discussed in a different context in [5]. In this note we will discuss 0-branes in the context of the derived category point of view [6]–[11]. We do this because we believe it is the most intrinsically "stringy" way of thinking about a D-brane. The derived category point

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of view springs directly from topological field theory. It is therefore superior to, say, using an effective D-brane world-volume action which presupposes a geometrical interpretation of the D-brane as a subspace. As the derived category picture highlights, this latter notion is awkward in many situations.

The basic question which emphasizes the rôle of 0-branes is the following. If I gave you all the intrinsic worldsheet information about a field theory based on a sigma model (with boundaries), could you construct the topology of the target space? Using the ideas of Bondal and Orlov [12] we will discuss to what extent this is true. Of course, the question should really be asked with "the target space" replaced by "a target space" as it is known that frequently two different target spaces can produce the same worldsheet theory. The central idea is to find which D-brane-like objects can be used as 0-branes. Unfortunately we will not find an answer to this important question but we can give some partial criteria.

In [13] it was shown that a flop between two topologically inequivalent target spaces X and X' is barely noticeable from the worldsheet point of view. As one passes through the flop, the worldsheet theory is generically smooth even though the target space acquires singularities. In this sense it seems much more natural to talk about birational classes of target spaces rather than topological classes. This is all in line with the general principle that a conformal field theory is an algebraic construction and so algebraic geometry should be the natural setting for stringy geometry.

In this note we will show how the behaviour of 0-branes *does* jump discontinuously as one passes through the flop. This is based on the ideas of Bridgeland [14] and the notion of Π-stability [15]–[11]. This issue was also noted in [7]. In addition the flop has been analyzed from the point of view of the D-brane world-volume in [17]. This example vividly illustrates how 0-branes *put the topology back into stringy geometry.*

For a particular choice of B field the conformal field theory becomes singular as one passes through the flop. It turns out that the derived category description of the flop depends on which side of this special point one passes. To fully understand this we will discuss monodromy around this point. This will also produce many more candidate 0-branes further complicating the picture.

2. Building a topological space

The data we choose to analyze is that of the topological B-model with boundaries. That is, we pretend that we have somehow extracted from the worldsheet data a knowledge of all possible boundaries (consistent with the topological field theory) and the corresponding topological field theory of open strings stretched between these boundaries. This data amounts to a knowledge of the bounded derived category of coherent sheaves on X [7, 9].[1]

Although using the derived category might at first be seen as an unnecessary foray into obscure mathematics, it is in fact a very natural construction from the point of view of topological field theory. No matter how much one might try to avoid the derived category,

[1] We are ignoring the effects of string field theory. It is quite likely that the complete picture requires an A_∞ category as discussed in [6, 18, 19, 8].

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anyone wishing to describe the topological sector of D-branes and type II strings would be forced to reinvent the subject or something very similar.

The basic data we have at our disposal consists of the following:

1. *Objects:* Classification of all possible topological B-type D-branes[2] associated with worldsheet boundaries.

2. *Morphisms:* The finite-dimensional "Hilbert Space" of open strings (with ghost number zero) with given D-branes at each end.

We will use the letters $\mathsf{A}, \mathsf{B}, \ldots$ to refer to objects and $\mathrm{Hom}(\mathsf{A}, \mathsf{B})$ etc. to refer to the spaces of morphisms. The joining of open strings along a common boundary gives the multiplication law in the operator algebra for open strings. In category language this corresponds to composition of morphisms:

$$\mathrm{Hom}(\mathsf{A}, \mathsf{B}) \times \mathrm{Hom}(\mathsf{B}, \mathsf{C}) \to \mathrm{Hom}(\mathsf{A}, \mathsf{C})\,. \tag{2.1}$$

As explained in [7, 9], the ghost number of the topological field theory can be extended to the D-branes. We may take an object A and produce a new object $\mathsf{A}[n]$ by shifting the ghost numbers associated to A by n. In terms of complexes, $\mathsf{A}[n]$ is the complex A shifted, or "translated", left n places. The open strings of ghost number n with boundaries A and B are then given by $\mathrm{Hom}^n(\mathsf{A}, \mathsf{B}) = \mathrm{Hom}(\mathsf{A}, \mathsf{B}[n]) = \mathrm{Hom}(\mathsf{A}[-n], \mathsf{B})$.

For the raw category, this is the only information we have. A D-brane is just an object in the category. The only information we have about it is the vector space of morphisms from it to any other D-brane. How then might we decide which D-branes correspond to 0-branes?

This question was answered by Bondal and Orlov [12] in some cases where X is not a Calabi-Yau. Suppose the canonical divisor K of X is such that $K.C > 0$ for every algebraic curve $C \subset X$, or alternatively suppose $K.C < 0$ for every algebraic curve $C \subset X$. Let us call an X satisfying either of these constraints an "anti-Calabi-Yau".

The open strings in Witten's B-model [22] on which the derived category construction is based are oriented. In particular $\mathrm{Hom}(\mathsf{A}, \mathsf{B})$ need not equal $\mathrm{Hom}(\mathsf{B}, \mathsf{A})$. Having said that, one might try to look for a symmetry which reverses the orientation of the strings. This would be given by a functor S such that

$$\mathrm{Hom}(\mathsf{A}, \mathsf{B}) \cong \mathrm{Hom}(\mathsf{B}, S\mathsf{A})^* \tag{2.2}$$

for all pairs A, B. We also demand that these isomorphisms preserve the structure of the category (we refer to [12] for details). Such a functor was called a "Serre functor" by Bondal and Orlov and shown to be unique if it exists.

Given an algebraic variety X as a target space, the derived category $\mathbf{D}(X)$ will have a Serre functor given by Serre duality:

$$S\mathsf{A} = \mathsf{A} \otimes K[d]\,, \tag{2.3}$$

where d is the dimension of X.

[2] We refer to [20, 21] for a full discussion of what is meant exactly by "B-type D-branes".

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Following Bondal and Orlov we define an object P in $\mathbf{D}(X)$ to be a "BO-point" object of codimension s if there is a Serre functor S and

$$\begin{array}{ll} \text{i)} & S\mathsf{P} \simeq \mathsf{P}[s] \\ \text{ii)} & \mathrm{Hom}^{<0}(\mathsf{P},\mathsf{P}) = 0 \\ \text{iii)} & \mathrm{Hom}^{0}(\mathsf{P},\mathsf{P}) = \mathbb{C}. \end{array} \tag{2.4}$$

The idea is that if we know X, then sky-scraper sheaves $\mathscr{O}_x$ are indeed BO-points in $\mathbf{D}(X)$. Furthermore, if X is an anti-Calabi-Yau then all BO-point objects must be of the form $\mathscr{O}_x[n]$ for some $x \in X$ and $n \in \mathbb{Z}$. The codimension s is equal to the dimension of X. Thus one can identify the points of X purely from their categorical description. This identification will fail for the case of Calabi-Yau's but for the meantime let us assume that we have determined these "point objects" for X.

Not only can we determine the set of points on X but we may also determine X as a topological space. To do this it is necessary to describe the set of open sets on X. The first step in this process is to identify elements of $\mathbf{D}(X)$ which correspond to (a complex with a single nonzero term given by) a line bundle. If we have already identified the points as above then this is easy. An object L is a "line-bundle object" if there is an $s \in \mathbb{Z}$ (which may depend on P) such that

$$\begin{array}{ll} \text{i)} & \mathrm{Hom}^{s}(\mathsf{L},\mathsf{P}) = \mathbb{C} \\ \text{ii)} & \mathrm{Hom}^{t}(\mathsf{L},\mathsf{P}) = 0 \quad \text{for } t \neq s\,, \end{array} \tag{2.5}$$

for all point objects P. Again Bondal and Orlov showed that if X exists then such line bundle objects correspond exactly to line bundles possibly translated in the complex by an arbitrary integer.

It is useful to fix the arbitrary translation symmetry in $\mathbf{D}(X)$. Choose a point object P_0. Next choose any line bundle object L_0 which satisfies (2.5) for $s=0$ with respect to P_0. A point object P will then be called an "absolute" point object if $\mathrm{Hom}^0(\mathsf{L}_0,\mathsf{P}) = \mathbb{C}$. This effectively restricts attention to point objects corresponding to sky-scraper sheaves at a fixed position in the complex. The set of absolute point objects is now in one-to-one correspondence with points on X. One may similarly define "absolute" line bundle objects as line bundle objects L satisfying $\mathrm{Hom}^0(\mathsf{L},\mathsf{P}_0) = \mathbb{C}$.

Consider now a morphism $\alpha \in \mathrm{Hom}(\mathsf{L}_1,\mathsf{L}_2)$ for an arbitrary pair $\mathsf{L}_1,\mathsf{L}_2$ of absolute line bundle objects. This induces a map

$$\alpha^*_{\mathsf{P}} : \mathrm{Hom}(\mathsf{L}_2,\mathsf{P}) \to \mathrm{Hom}(\mathsf{L}_1,\mathsf{P}) \tag{2.6}$$

for any object P in the obvious way. Let U_α be the set of absolute point objects in $\mathbf{D}(X)$ such that $\alpha^*_{\mathsf{P}} \neq 0$. The geometrical meaning of U_α is as follows. α is a global map from L_1 to L_2 and is thus equivalent to a section of $\mathsf{L}_1^\vee \otimes \mathsf{L}_2$. A divisor D_α can be associated to the zeroes of this section in the usual way. If α is chosen suitably, this divisor corresponds to a codimension one subspace of X. U_α is then the open set in X given be the complement of D_α. These open sets U_α can be used to generate a topology (namely the Zariski topology) for X.

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3. Hunting the 0-brane

If we could do the same when X is a Calabi-Yau then we would be done. The Serre functor in this case is given by $S\mathsf{A} = \mathsf{A}[d]$, where d is the dimension of X. This may be seen directly at the worldsheet level. One can change the orientation of the open strings by reversing the sign of the ghost number. Then use a spectral flow argument along the lines of [23] to shift the ghost numbers by d. This gives the desired Serre functor as claimed.

The problem for the Calabi-Yau case is that condition i) in (2.4) is trivial thus weakening the criteria. There are many objects in $\mathbf{D}(X)$ which satisfy the BO-point condition without being sky-scraper sheaves. We therefore need a more stringent condition for being called a "point object". If x and y are points in X then one may show that[3]

$$\begin{aligned} \dim \mathrm{Hom}^q(\mathscr{O}_x, \mathscr{O}_x) &= \binom{d}{q} \\ \mathrm{Hom}^q(\mathscr{O}_x, \mathscr{O}_y) &= 0 \quad \text{for } x \neq y\,. \end{aligned} \tag{3.2}$$

This motivates the following. An object A in $\mathbf{D}(X)$ satisfying $\dim \mathrm{Hom}^q(\mathsf{A},\mathsf{A}) = \binom{d}{q}$ will be called a "pre-point object". Two non-isomorphic pre-point objects A and B will be called "mutually consistent" if they satisfy $\dim \mathrm{Hom}^q(\mathsf{A},\mathsf{B}) = 0$ for any q.

Unfortunately there are many more pre-point objects in $\mathbf{D}(X)$ than there are points. It is not hard to see in any example that many pairs of pre-point objects will not be consistent in the above sense. Let us consider how we can further restrict the conditions on a object being declared a point.

As an example consider the case where X is a quintic hypersurface in $\mathbb{P}^4$. Monodromy around loops in the moduli space of complexified Kähler forms is believed to induce autoequivalences of $\mathbf{D}(X)$ in accordance with Kontsevich's homological mirror symmetry conjecture. See [6, 24, 25], for example, for an account of this. In particular monodromy around the "conifold" point in the case of the quintic is obtained from an autoequivalence given by a particular Fourier-Mukai transform [26, 24, 27]. Under such a transform, the sky-scraper sheaf of a point $\mathscr{O}_x$ is transformed into $\mathscr{I}_x[1]$ — the ideal sheaf of functions vanishing at the same point which is then shifted one place left in the complex.

It is therefore impossible to declare that $\mathscr{O}_x$ has any more right to be associated to a point than $\mathscr{I}_x[1]$ when only looking at intrinsic categorical data. Lack of mutual consistency can be seen from the fact that $\mathrm{Hom}^1(\mathscr{I}_x[1], \mathscr{O}_y) = \mathbb{C}$ for any $y \neq x$.

One possibility which should immediately spring to mind as regards fixing this problem is that of "D-brane charge". We may restrict our collection of candidate points by insisting that they all have the same D-brane charge.

For the derived category of coherent sheaves, an object

$$\cdots \to \mathscr{E}^0 \to \mathscr{E}^1 \to \mathscr{E}^2 \to \cdots\,, \tag{3.3}$$

[3] For this and many subsequent calculations the following spectral sequence is useful. Let $i: Z \to X$ be a smooth embedding with N the normal bundle. Then if $\mathscr{A}, \mathscr{B}$ are sheaves on Z

$$E_2^{p,q} = \mathrm{Hom}^p_{\mathscr{O}_Z}(\mathscr{A}, \mathscr{B} \otimes \Lambda^q N) \Rightarrow \mathrm{Hom}^{p+q}_{\mathbf{D}(X)}(i_*\mathscr{A}, i_*\mathscr{B}). \tag{3.1}$$

will have a D-brane charge given by the Chern character[4]

$$ch(\mathscr{E}^\bullet) = \sum_n (-1)^n \, ch(\mathscr{E}^n)\,. \tag{3.4}$$

The intrinsic categorical information knows about the Chern character because of the Hirzebruch-Riemann-Roch theorem:

$$\sum_n (-1)^n \dim \mathrm{Hom}(\mathsf{A}, \mathsf{B}[n]) = \int_X ch(\mathsf{A})\, ch(\mathsf{B})^\vee \, td(T_X)\,. \tag{3.5}$$

This gives a skew-symmetric inner product on the derived category which is believed to be mirror to the intersection form of 3-cycles [28, 21, 29]. As such it should be non-degenerate. Also, assuming the Hodge conjecture to be true, the Chern characters of all objects in $\mathbf{D}(X)$ span $H^{\mathrm{even}}(X, \mathbb{Q})$. We may therefore claim the following: If A and A' are two objects in $\mathbf{D}(X)$ and

$$\sum_n (-1)^n \dim \mathrm{Hom}(\mathsf{A}, \mathsf{B}[n]) = \sum_n (-1)^n \dim \mathrm{Hom}(\mathsf{A}', \mathsf{B}[n])\,, \tag{3.6}$$

for all objects B in $\mathbf{D}(X)$, then A and A' have the same Chern character and therefore D-brane charge. Thus we know how to analyze D-brane charge from purely categorical data.

The approach one might try to take could then be as follows. Choose a pre-point object in $\mathbf{D}(X)$. Now catalogue all other pre-point objects in $\mathbf{D}(X)$ having the same D-brane charge. In simple cases it is perhaps conceivable that one might obtain a *mutually consistent* set of pre-point objects. The pre-point objects would hopefully therefore be fully accounted for by all the points on X and their translations in $\mathbf{D}(X)$ by an even number. Unfortunately this procedure is doomed to failure in some (if not all) cases as we now demonstrate.

4. The flop

Suppose X contains a rational curve C which may be contracted down to a point by varying the Kähler form. We may then "proceed through" this wall of the Kähler cone to produce a flop of X into another Calabi-Yau space X' with a new corresponding rational curve C'. In general X' can be expected to be topologically distinct from X.

The topological B-model is insensitive to any change in the Kähler form and so $\mathbf{D}(X)$ must remain invariant under such a process. Indeed, it was shown explicitly in [30, 14] that $\mathbf{D}(X)$ is equivalent to $\mathbf{D}(X')$. This presents a challenge to our program. If we are to construct X from $\mathbf{D}(X)$ we should equally be allowed to construct X'.

Bridgeland [14] gave an explicit form to the equivalence between $\mathbf{D}(X)$ and $\mathbf{D}(X')$. Away from the curves C and C', the spaces X and X' are identical. That is, we have a one-to-one correspondence between points in these two spaces as shown schematically

[4] D-brane charge is measured by K-theory. By considering the Chern character we are only looking at the free part. The torsion part is accessible from the derived category and might be of interest in some cases.

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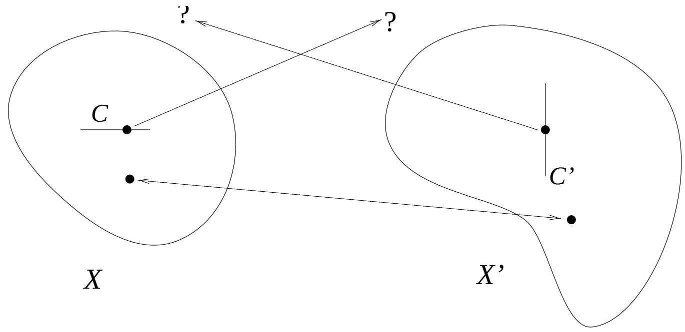

Figure 1: The correspondence between points in X and X'.

in figure 1. Naturally this correspondence breaks down for points within C and C'. A sky-scraper sheaf on a point in C corresponds to an object in $\mathbf{D}(X')$ which is not simply a sheaf but rather a longer complex.[5] Similarly a point $x' \in C'$ corresponds to a specific non-sheaf object $\mathsf{Z}_{x'}$ in $\mathbf{D}(X)$.

This object $\mathsf{Z}_{x'}$ is clearly a pre-point object and has the same D-brane charge as a point object in X. It does not however correspond to a point in X. We therefore need more information to rule it out as a point in X. Since the topological field theory cannot distinguish between X and X', we are forced to go beyond the topological field theory, and thus the intrinsic information content of the derived category.

We want to use the Π-stability of [15] as a criterion for ruling out $\mathsf{Z}_{x'}$. In the region of Kähler form moduli space corresponding roughly to the Kähler cone of X we will show that $\mathsf{Z}_{x'}$ is unstable. This was also alluded to in [7].

The data required for Π-stability is a "grading" $\varphi \in \mathbb{R}$ associated to a subset of the objects in $\mathbf{D}(X)$. This subset of objects is the Π-stable set of objects. The grading is determined mod 2 as the phase of the central charge of the corresponding D-brane and, as such, depends continuously on the complexified Kähler form $B + iJ$.

Note that the central charge of the D-brane is data visible in the non-compact 4-dimensional $N = 2$ supersymmetric field theory and thus fits into our game rules. It is data we are allowed to use to construct X without knowing anything about X beforehand.

The exact rules for Π-stability are rather convoluted as explained in [11]. As $B+iJ$ is varied, objects will enter and leave the set of stable objects according to their values of φ. The statement of Π-stability is rather simple if we focus attention on a basic set of objects $\{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ where we know C can potentially decay to $\bar{\mathsf{A}}$ (i.e., the anti-brane of A which is actually the same thing as $\mathsf{A}[1]$ as argued in [7]) and B. Assume $\bar{\mathsf{A}}$ and B are stable and we wish to determine if C is stable relative to decay into $\bar{\mathsf{A}}$ and B.

Such a triple of objects forms a "distinguished triangle" in $\mathbf{D}(X)$ and is written:

$$\begin{tikzcd} & \mathsf{C} \arrow[dl, "{[1]}"'] & \\ \mathsf{A} \arrow[rr] & & \mathsf{B}. \arrow[ul] \end{tikzcd} \tag{4.1}$$

[5]In fact it's a "perverse sheaf" [14].

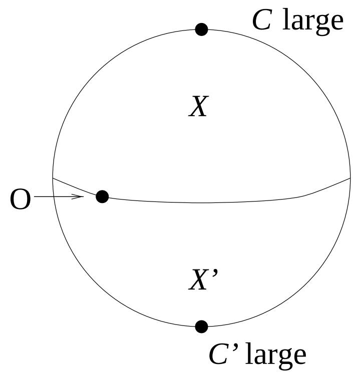

Figure 2: The moduli space of $B + iJ$ for the flop.

The "[1]" denotes that the morphism is really from C to $\mathsf{A}[1]$. C is then stable with respect to A and B if and only if $\varphi(\mathsf{B}) - \varphi(\mathsf{A}) < 1$.

It is important to note that the "[1]" in (4.1) can be shuffled around to any edge. For example, (4.1) can be rewritten

$$\begin{tikzcd} & \mathsf{C} \arrow[dl] & \\ \mathsf{A}[1] \arrow[rr, "{[1]}"] & & \mathsf{B}. \arrow[ul] \end{tikzcd} \tag{4.2}$$

This symmetry between the edges means that if the difference in φ's exceeds one on any of the edges of the triangle, the object in the opposite vertex will decay. This symmetry is one of the important properties of the derived category which cannot be reproduced in any "abelian" category such as the category of vector bundles, and shows why categories such as the latter cannot fully model D-brane decay [11].

In order to analyze the flop we use the following D-branes as building blocks:

1. The skyscraper sheaf $\mathscr{O}_x$ for $x \in C$.

2. The skyscraper sheaf $\mathscr{O}_y$ for $y \notin C$.

3. The structure sheaf $\mathscr{O}_C$ of the flopping curve, i.e., the 2-brane wrapped around C.

The periods of these D-branes are very easy to compute. We can use mirror symmetry to compute periods exactly (with respect to α'-corrections) as in [31, 11]. We may localize the picture by assuming that all curves in X are very large except for those homologous to C. The required Picard–Fuch's differential equation was then determined in section 5.4 of [32].

The part of the moduli space of $B + iJ$ which concerns us can be viewed as a 2-sphere as shown in figure 2. The large radius limit of $C \subset X$ is put at one pole and the large radius limit of $C' \subset X'$ is put at the other pole. On the equator, which separates the X "phase" from the X' "phase" we have a point where the worldsheet theory becomes singular. This latter point will be viewed as the origin in our final coordinate system, so we label it O.

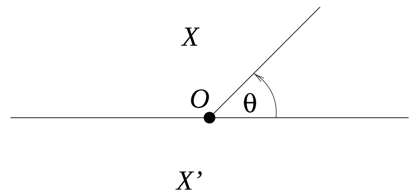

Figure 3: The complex t-plane.

There is a natural "algebraic" coordinate z on this sphere with $z = 0$ in the large C limit, $z = \infty$ in the large C' limit and $z = 1$ at O. In terms of this coordinate z we have a Picard–Fuchs equation

$$\left(z\frac{d}{dz}\right)^2 \Phi - z\left(z\frac{d}{dz}\right)^2 \Phi = 0\,. \tag{4.3}$$

This has a general solution

$$\Phi = C_1 + C_2 \log(z)\,. \tag{4.4}$$

The rules of the "mirror map" [33] (see [34] for more details) then tell us

$$t = \int_C B + iJ = \frac{1}{2\pi i}\frac{\Phi_1}{\Phi_0}\,, \tag{4.5}$$

where Φ_0 is the period which is regular and equal to 1 at the large C limit (i.e., $z \to 0$ limit) and Φ_1 is the period whose limiting behaviour is $\log(z)$ for $z \to 0$. Clearly from (4.3) we have

$$t = \frac{1}{2\pi i}\log(z)\,. \tag{4.6}$$

Thus $t = 0$ at O as promised.

In order to compute central charges we use [35, 36][6]

$$Z(\mathsf{A}) = \int_X e^{B+iJ}\, ch(\mathsf{A})\sqrt{td(T_X)} + \text{quantum corrections}\,, \tag{4.7}$$

where the quantum corrections are determined from the fact that $Z(\mathsf{A})$ is a period. This immediately yields:

$$\begin{aligned} Z(\mathscr{O}_x) &= Z(\mathscr{O}_y) = 1 \\ Z(\mathscr{O}_C) &= t\,, \end{aligned} \tag{4.8}$$

i.e., there are no quantum corrections! This makes the flop particularly easy to analyze. In most other examples the complexity of the Picard-Fuchs system makes the formula for central charged much less amenable as seen in [11].

We may capture all the relevant properties of D-brane behaviour near the flop by assuming $t = \epsilon e^{i\theta}$ where $\epsilon \ll 1$. We thus parametrize the moduli space as in figure 3.

[6]Note that we are using the sign convention for B from section 3 rather than section 2 of [11]!

The grades of our D-branes can be determined from $\varphi = -(1/\pi)\arg(Z)$ where at large radius limit (i.e., $\theta = \pi/2$) we insist that $-2 < \varphi \le 0$ [11]. Thus

$$\begin{aligned} \varphi(\mathscr{O}_x) &= \varphi(\mathscr{O}_y) = 0 \\ \varphi(\mathscr{O}_C) &= -\frac{\theta}{\pi}\,. \end{aligned} \tag{4.9}$$

We may now compute D-brane stability. The first distinguished triangle of interest is

$$\begin{tikzcd} & \mathsf{A} \arrow[dl, "{[1]}", "1-\frac{\theta}{\pi}"'] & \\ \mathscr{O}_C \arrow[rr, "\frac{\theta}{\pi}"'] & & \mathscr{O}_x \arrow[ul, "0"'] \end{tikzcd} \tag{4.10}$$

Here we label each edge of the triangle with the difference in grading between the vertices. It is whether this label is greater than or less than one which determines the stability of the opposite vertex. In derived category language A is the "Cone" of the natural map $\mathscr{O}_C \to \mathscr{O}_x$. The short exact sequence

$$0 \to \mathscr{O}_C(-1) \to \mathscr{O}_C \to \mathscr{O}_x \to 0 \tag{4.11}$$

tells us that A is given by $\mathscr{O}_C(-1)[1]$. The triangle (4.10) dictates that $\mathscr{O}_x$ will decay into A and $\mathscr{O}_C$ if $\theta < 0$. That is the 0-brane $\mathscr{O}_x$ becomes unstable as we pass from the X phase into the X' phase through $\theta = 0$. *This is exactly what we wanted!* If we impose the stability constraint on our 0-branes then we lose the "bogus" ones living on a curve flopped out of existence.

Note that there is *no* nontrivial homomorphism $\mathscr{O}_C \to \mathscr{O}_y$ with $y \notin C$. This means that there is no distinguished triangle of the form (4.10) with x replaced by y. This prevents the 0-branes $\mathscr{O}_y$, away from the flop, from decaying — again in line with expectation.

We would now like to find the new 0-branes which jump into existence as we pass into the X' phase. It would be nice to have a systematic way of building triangles relevant to our discussion. Clearly homomorphisms from $\mathscr{O}_C$ to something else will be of interest since the relative grading is likely to change as we encircle O. We might therefore consider a map $\mathscr{O}_C \to \mathsf{A}$ next. Using the spectral sequence (3.1) we can show that $\mathrm{Hom}(\mathscr{O}_C, \mathsf{A}) = 0$ so we get nothing new here. As a next attempt one can show that $\mathrm{Hom}(\mathscr{O}_C[-1], \mathsf{A}) = \mathbb{C}^2$. Let us see what happens when we consider the cone B of a map $\mathscr{O}_C[-1] \to \mathsf{A}$. This generates the triangle

$$\begin{tikzcd} & \mathsf{B} \arrow[dl, "{[1]}", "-\frac{\theta}{\pi}"'] & \\ \mathscr{O}_C[-1] \arrow[rr, "1+\frac{\theta}{\pi}"'] & & \mathsf{A} \arrow[ul, "0"'] \end{tikzcd} \tag{4.12}$$

Thus B jumps into existence as θ becomes negative, i.e., we pass into the X' phase. It is not hard to show that B is a pre-point object in the sense of (3.2) and has the same D-brane charge as a point. Since $\mathrm{Hom}(\mathscr{O}_C[-1], \mathsf{A}) = \mathbb{C}^2$ we had a choice of which map

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to use to give B. This essentially gives us a whole $\mathbb{P}^1$'s-worth of objects B. Indeed B is exactly the object in $\mathbf{D}(X)$ studied by Bridgeland in [14] where he showed that the $\mathbb{P}^1$ parametrized by the B objects fits nicely into $X - C$ to produce precisely X'. That is, he showed rigorously that the objects B really are the point objects for points on $C' \subset X'$, and so the objects B are the $\mathsf{Z}_{x'}$ objects discussed earlier in this section.

It is very gratifying to note that the objects selected by Bridgeland for this purpose are exactly the ones selected by Π-stability for this same rôle.

Let us now continue this process. Since $\mathrm{Hom}(\mathscr{O}_C[-1], \mathsf{B}) = \mathbb{C}$, we may construct another distinguished triangle:

$$\begin{tikzcd} & \mathsf{C} \arrow[dl, "-\frac{\theta}{\pi}", "{[1]}"'] & \\ \mathscr{O}_C[-1] \arrow[rr, "1+\frac{\theta}{\pi}"'] & & \mathsf{B}, \arrow[ul, "0"'] \end{tikzcd} \tag{4.13}$$

which shows that another D-brane, C, comes into existence along with B as we pass into the X' phase. Lastly $\mathrm{Hom}(\mathscr{O}_C[-1], \mathsf{C}) = 0$ but $\mathrm{Hom}(\mathscr{O}_C[-2], \mathsf{C}) = \mathbb{C}^2$ so the triangle

$$\begin{tikzcd} & \mathsf{D} \arrow[dl, "-1-\frac{\theta}{\pi}", "{[1]}"'] & \\ \mathscr{O}_C[-2] \arrow[rr, "2+\frac{\theta}{\pi}"'] & & \mathsf{C} \arrow[ul, "0"'] \end{tikzcd} \tag{4.14}$$

brings a new object D into existence as θ falls below $-\pi$, i.e., after we circle O clockwise in figure 3 and come back into the X phase. Note that since $\mathrm{Hom}(\mathscr{O}_C[-2], \mathsf{C}) = \mathbb{C}^2$, we actually have a $\mathbb{P}^1$'s-worth of objects D. We claim that the D objects are the new objects representing points on C after monodromy around O. This is exactly the mechanism for monodromy presented in the context of the quintic in [11] except that here we have had to pass through a number of triangles to perform the monodromy. Note also that even though D and $\mathscr{O}_x$ are quite different objects in $\mathbf{D}(X)$, they have the same D-brane charge. Thus this monodromy cannot be seen purely at the level of cohomology classes.

Note that this monodromy is consistent with the Fourier–Mukai transform predicted by Horja [24, 37] and Seidel and Thomas [27]. That is, in the language of [27]:

$$\mathsf{D} = \mathrm{Cone}(\mathrm{hom}(\mathscr{O}_C, \mathscr{O}_x) \overset{\mathbf{L}}{\otimes} \mathscr{O}_C \to \mathscr{O}_x)\,. \tag{4.15}$$

In order to further clarify this monodromy statement, let us consider what happens if we try to perform the flop by moving *counter-clockwise* around O to leave the X phase and pass into the X' phase. Now we have the following relevant distinguished triangle

$$\begin{tikzcd} & \mathscr{O}_x \arrow[dl, "1-\frac{\theta}{\pi}", "{[1]}"'] & \\ \mathscr{O}_C \arrow[rr, "\frac{\theta}{\pi}"'] & & \mathscr{O}_C(1), \arrow[ul, "0"'] \end{tikzcd} \tag{4.16}$$

which shows how $\mathscr{O}_x$ becomes unstable as $\theta > \pi$. We claim then the triangle (4.14) is exactly the triangle (4.16) under clockwise monodromy around O, i.e., $\mathscr{O}_C$ becomes $\mathscr{O}_C[-2]$ and $\mathscr{O}_C(1)$ becomes C. This is again consistent with the Fourier-Mukai transform above.

5. Discussion

The derived category picture emphasizes the issues involved in declaring a given D-brane to be a point, i.e., a 0-brane. The occurrence of flops and monodromy show that many objects in the derived category have a right to be called a point, but only in a particular region of Kähler moduli space.

In any sufficiently complex example it is clear that even if one restricts to a specific D-brane charge, there are an infinite number of candidate point objects for each point in X. Many of these are related by monodromy transforms like the one we saw above for a flop. One can also generate similar monodromy transforms for points on the exceptional divisor of an orbifold. Insisting that a point object be *stable* removes all the monodromy images of each point.

It is overly-optimistic to assume that we have given enough conditions to cut down the number of pre-point objects to be left with the consistent set of true point objects. Understanding this further is key to knowing how to extract target space topology from worldsheet information.

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